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Relationship between the energy eigenstates of Calogero–Sutherland models with oscillator and coulomb-like potentials

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Abstract. We establish a simple algebraic relationship between the energy eigenstates of the rational Calogero–Sutherland model with harmonic oscillator and coulomb-like potentials. We show that there is an underlying $SU(1,1)$ algebra in both of these models which plays a crucial role in such an identification. Furthermore, we show that our analysis is in fact valid for any many-particle system in arbitrary dimensions whose potential term (apart from the oscillator or the coulomb-like potential) is a homogeneous function of coordinates of degree -2. The explicit coordinate transformation which maps the coulomb-like problem to the oscillator one has also been determined in some specific cases.

1. Introduction

The rational Calogero–Sutherland model (CSM) describes a system of N particles interacting with each other via a long-range inverse square interaction [1–3] which are confined on a line by a simple harmonic oscillator (SHO) potential. This model is exactly solvable and the spectrum as well as the eigenfunctions are well known. Furthermore, it is known that the rational CSM, with the SHO potential replaced by a coulomb-like interaction, is also exactly solvable [4]. The remarkable common feature of both the models is that they reduce to the usual harmonic oscillator or the coulomb-like problem in dimensions greater than one, once the short distance correlations are factored out.

It is worth pointing out that the only two problems which can be solved for all partial waves in dimensions greater than one are the usual harmonic oscillator and the coulomb problems. Furthermore, a mapping relating the energy eigenvalues as well as the eigenfunctions of these two models exists in any number of dimensions [5, 6]. It is then natural to enquire if there is a mapping between the energy eigenvalues as well as eigenfunctions of the rational CSM and the same quantities of the CSM with the coulomb-like interaction.

The purpose of this paper is to show that such a mapping between these two types of CSM indeed exists. In particular, we show that both the models possess an underlying $SU(1,1)$ algebra with different realizations for the generators of the algebra, much akin to the usual harmonic oscillator or the coulomb problem [5, 6]. Using this underlying algebra, we show that the energy eigenvalues as well as the eigenfunctions of the rational CSM with the coulomb-like

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interaction can be obtained from the corresponding CSM oscillator problem. Our results are valid for all types of rational CSM, namely, the CSM associated with the root structure of A_N, B_N, C_N, BC_N and D_N. Thus, we are able to generalize the A_N type of CSM with coulomb-like interaction [4] to BC_N, B_N, C_N and D_N type and hence show that all these models are also exactly solvable. Thus we are adding new members to the family of the exactly solvable one-dimensional many-body systems.

We also generalize these results to several higher-dimensional Calogero–Sutherland types of models. In particular, we show that such a mapping is possible in any arbitrary dimension provided the long-range many-body interaction of these models, like its one-dimensional counterpart, is a homogeneous function of the coordinates with degree -2.

The plan of the paper is as follows. In section 2, the mapping between the SHO and the coulomb-like CSM problems is established through an underlying $SU(1,1)$ algebra which is shown to exist in both the problems. In particular, in section 2.1, we discuss the underlying $SU(1,1)$ algebra in the CSM with the coulomb-like potential. In section 2.2, a similar algebraic structure of the many-body systems with the SHO potential is presented. The mapping between the two is established in section 2.3. In section 3, we discuss the explicit coordinate transformation which maps one problem onto the other. We find a set of coupled second-order nonlinear differential equations, the solution of which determines the explicit form of the coordinate transformation. We also solve this differential equation for some specific many-particle systems. Discussions have been made in section 4 regarding the higher-dimensional generalization of the mapping relating these two types of Hamiltonians. Finally, in section 5, we summarize the results obtained in this paper and point out some of the open problems. In appendix A, we present the energy spectrum and some of the eigenfunctions of the coulomb-like CSM of B_N type. In appendix B, we show that the Casimir operator of the $SU(1,1)$ group is the angular part of the CSM Hamiltonian corresponding to the coulomb-like or the oscillator problems. We also indicate here how the group property enables us to use the method of separation of variables.

2. The mapping

2.1. Algebra of the coulomb-like problem

Let us consider the Hamiltonian ($\hbar = m = 1$),

$$H_{\rm c} = -\frac{1}{2}\triangle_x + V(x_1, \ldots, x_N) - \frac{\alpha}{x} \tag{1}$$

where

$$x = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2} \qquad \triangle_x = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}. \tag{2}$$

The coordinates of the N particles are denoted by x_i in (2). We fix the convention that all Roman indices run from 1 to N while all Greek indices run from 1 to N'. The many-body interaction $V(x)$ in (1) is a homogeneous function of degree -2. In particular,

$$\sum_{i=1}^{N} x_i \frac{\partial V}{\partial x_i} = -2V. \tag{3}$$

It may be noted that the potential term V of the rational CSM of A_n type,

$$V_{A_n}(\{x_i\}) = \frac{g}{2}\sum_{i<j}(x_i - x_j)^{-2} \tag{4}$$

indeed satisfies this condition. In fact the long-range interaction terms of the rational BC_N, B_N, C_N, D_N type CSM also satisfy this condition. In particular,

$$V_{BC_N}(g_1, g_2, g_3) = \frac{g_1}{2}\sum_{i<j}[(x_i - x_j)^{-2} + (x_i + x_j)^{-2}] + g_2\sum_i x_i^{-2} + \frac{g_3}{2}\sum_i x_i^{-2} \tag{5}$$

$V_{B_n}(= V_{BC_n}(g_1, g_2, g_3 = 0))$, $V_{C_n}(= V_{BC_n}(g_1, g_2 = 0, g_3))$ and $V_{D_n}(= V_{BC_n}(g_1, g_2 = 0, g_3 = 0))$ have the property (3). Unless mentioned otherwise, throughout this paper we consider arbitrary $V(x)$ satisfying the property (3) even though schematically we write it as $V(x)$.

Let us define the operators k_1, k_2, k_3 as

$$\begin{aligned} k_1 &= \frac{1}{2}(x\triangle_x - 2xV(x) + x) \\ k_2 &= \mathrm{i}\left(\frac{N-1}{2} + \sum_i x_i\frac{\partial}{\partial x_i}\right) \\ k_3 &= -\frac{1}{2}(x\triangle_x - 2xV(x) - x). \end{aligned} \tag{6}$$

It is easily shown that these three operators constitute a $SU(1,1)$ algebra, namely,

$$[k_1, k_2] = -\mathrm{i}k_3 \qquad [k_2, k_3] = \mathrm{i}k_1 \qquad [k_3, k_1] = \mathrm{i}k_2. \tag{7}$$

Let us emphasize again that the $SU(1,1)$ algebra as given here in terms of the generators k_1, k_2 and k_3 is valid for any V satisfying equation (3). We now show that the eigenvalue equation for the Hamiltonian as given by equation (1) can also be written as an eigenvalue equation for the generator of $SU(1,1)$. To see this, note the following identity:

$$(k_1 + k_3)H_{\mathrm{c}} = -\tfrac{1}{2}(k_1 - k_3) - \alpha. \tag{8}$$

Now, following the standard procedure [7] and with the help of equation (8), the eigenvalue equation

$$H_{\mathrm{c}}|N, M\rangle = E_M|N, M\rangle \tag{9}$$

can be written as

$$\left[k_3 - \frac{\alpha}{\sqrt{-2E_M}}\right]\mathrm{e}^{\mathrm{i}k_2\theta_M}|N, M\rangle = 0 \tag{10}$$

where the function θ_M is defined by

$$\cosh\theta_M = \frac{1 - 2E_M}{\sqrt{-8E_M}} \qquad \sinh\theta_M = -\frac{1 + 2E_M}{\sqrt{-8E_M}}. \tag{11}$$

Thus, the eigenvalue equation for H_{c} has been transformed into an eigenvalue equation for the generator k_3. The eigenvector $|N, M\rangle$ with the eigenvalue E_M in (9) is defined to characterize the N particle state with M as the principal quantum number. In general, M can be expressed as a sum of different non-negative integers to characterize the degenerate states, depending on the particular form of $V(x)$. Even though we do not address here the question of degeneracy of the many-body system, it should be noted that the eigenvectors $|N, M\rangle$ do not span the whole eigenspace of H_{c}. In particular, the eigenstates $|N, M\rangle$ transform under the unitary irreducible representations of $SU(1,1)$ labelled by a real constant $\phi(< 0)$, where ϕ is related to the eigenvalue q of the Casimir operator as $q = \phi(\phi + 1)$. Thus, $|N, M\rangle$ belongs to the $SU(1,1)$ orbit of the ground state $|N, M; \phi = \phi_0\rangle$, where ϕ_0 denotes the minimum admissible value of ϕ. As shown in appendix B, the energy eigenvalue E_M is determined in terms of the Casimir operator of $SU(1,1)$ as

$$E_{m,q} = -\frac{\alpha^2}{2}\left[m + \frac{1}{2} + \left(q + \frac{1}{4}\right)^{1/2}\right]^{-2} \tag{12}$$

where m is a non-negative integer and q is the eigenvalue of the Casimir operator.

2.2. Algebra of the oscillator problem

Let us consider the Hamiltonian ($\hbar = m = 1$),

$$H_{\mathrm{sho}} = \tfrac{1}{2}(-\triangle_y + y^2 + 2V(y)) \tag{13}$$

where

$$\triangle_y = \sum_{\mu=1}^{N'} \frac{\partial^2}{\partial y_\mu^2} \qquad y^2 = \sum_{\mu=1}^{N'} y_\mu^2. \tag{14}$$

The potential $V(y)$ is again a homogeneous function of y with degree -2, i.e. it satisfies a condition analogous to equation (3).

We now define three operators k_1, k_2 and k_3 for the oscillator as follows:

$$\begin{aligned} k_1 &= \frac{1}{4}(\triangle_y + y^2 - 2V(y)) \\ k_2 &= \frac{\mathrm{i}}{4}\left(N' + 2\sum_\mu y_\mu \frac{\partial}{\partial y_\mu}\right) \\ k_3 &= \frac{1}{2}H_{\mathrm{sho}}. \end{aligned} \tag{15}$$

Note that these three operators again constitute a $SU(1,1)$ algebra and the Hamiltonian is proportional to k_3. As a result, the eigenvalue equation of the Hamiltonian is also the eigenvalue equation for the operator k_3. In particular,

$$H_{\mathrm{sho}}|N', M'\rangle = e_{M'}|N', M'\rangle \to k_3|N', M'\rangle = \tfrac{1}{2}e_{M'}|N', M'\rangle. \tag{16}$$

The eigenvector $|N', M'\rangle$ with the eigenvalue $e_{M'}$ in equation (16) is defined, as in the case of the coulomb problem in section 2.1, to characterize the N' particle state with M' as the principal quantum number. The eigenstates $|N', M'\rangle$ transform under the unitary irreducible representations of $SU(1,1)$, labelled by a real constant $\phi'(<0)$, where ϕ' is related to the eigenvalue q of the Casimir operator as, $q = \phi'(\phi' + 1)$. Thus, $|N', M'\rangle$ do not span the whole eigenspace of H_{sho}. Instead, it belongs to the $SU(1,1)$ orbit of the ground state $|N', M'; \phi' = \phi_0'\rangle$, where ϕ_0' denotes the minimum admissible value of ϕ'. We do not address the question of degeneracy in this paper. In appendix B, we again show that the energy eigenvalue $e_{M'}$ is determined in terms of the Casimir operator of $SU(1,1)$ as

$$e_{m',q} = 2m' + 1 + (1 + 4q)^{1/2} \tag{17}$$

where m' is a non-negative integer and q is the eigenvalue of the Casimir operator. We show in appendix B that different representations of the Casimir operator in terms of the generators (6) and (15) correspond to the angular part of H_{c} and H_{sho}, respectively (apart from a constant).

2.3. The relationship

In order to obtain the relationship between the eigenspectrum of the two CSM problems, we assume that the potentials $V(x)$ and $V(y)$ have the same functional dependence on the x and the y coordinates, respectively. However, the strength of the interaction may be different in the two cases which we do not mention here explicitly in order to avoid notational clumsiness.

We have considered two different representations for the generators of the $SU(1,1)$ algebra, given by (6) and (15). However, in both cases one is using the same positive discrete series representation of the $SU(1,1)$ algebra. Furthermore, in this representation, k_3 is taken to be diagonal in both cases. Thus, the isomorphism between the two sets of eigenvectors corresponding to two different representations of the generators of $SU(1,1)$ naturally follows.

Now note that both equations (10) and (16) are eigenvalue equations for k_3. Thus, on comparing these two equations, we have

$$|N', M'\rangle = \mathrm{e}^{\mathrm{i}k_2\theta_M}|N, M\rangle \qquad e_{M'} = \frac{\sqrt{2}\alpha}{\sqrt{-E_M}} \tag{18}$$

or,

$$|N, M\rangle = \mathrm{e}^{-\mathrm{i}k_2\theta_M}|N', M'\rangle \qquad E_M = -\frac{2\alpha^2}{(e_{M'})^2}. \tag{19}$$

This establishes the mapping between the eigenvalues as well as the eigenfunctions of H_{c} and H_{sho}. This also implies that H_{c} is exactly solvable provided H_{sho} is so and *vice versa*. Since this analysis is valid for any $V(x)$ satisfying equation (3), this means that we have found a class of new, exactly solvable, many-body problems in one dimension. For example, the B_N, C_N, D_N, BC_N CSM, with the harmonic oscillator potential replaced by the coulomb-like potential, must also be exactly solvable many-body problems. As an illustration, the eigenvalues as well as some of the eigenfunctions of the B_N-model with coulomb-like potential have been worked out in Appendix A.

The second relation in equation (18) as well as (19) describes the relationship between the energy spectra of the two problems. The fact that this relationship is indeed valid is easily checked by using equations (12) and (17) and identifying m as m'. Since, the eigenvalue q of the Casimir operator is independent of any particular representation of the generators (i.e. equation (6) or (15)), it is expected that a comparison of the known energy spectra of H_{c} and H_{sho} would in general relate different quantum numbers as well as parameters of a particular theory to the another. We work out here some known examples to explore such relations.

(a) Let us first consider a simple example, i.e. consider the potentials

$$V(x) = gx^{-2} \qquad V(y) = g'y^{-2}. \tag{20}$$

The energy eigenvalues E_m and e'_m for this choice of $V(x)$ and $V(y)$ are given by

$$E_{m,k} = -\tfrac{1}{2}\alpha^2(m + \tfrac{1}{2} + \lambda_k)^{-2} \qquad e_{m',k'} = 2m' + 1 + \lambda'_{k'} \tag{21}$$

where λ_k and $\lambda'_{k'}$ are defined as

$$\lambda_k = [\tfrac{1}{2}(2k + N - 2) + 2g]^{1/2} \qquad \lambda'_{k'} = [\tfrac{1}{2}(2k' + N' - 2) + 2g']^{1/2}. \tag{22}$$

One can easily see that equations (19)–(22) are consistent with each other provided the following relations hold

$$N' = 2(N - 1) \qquad g' = \frac{g}{4} \qquad k' = 2k \qquad m' = m. \tag{23}$$

We will see in the next section that the first two relations also follow from the coordinate transformation.

(b) Consider the rational CSM (with SHO) of A_n type and the corresponding coulomb-like problem [4]. In this case, the energy eigenvalues $E_{m,k}$ and $e_{m',k'}$ are given by

$$E_{m,k} = -\tfrac{1}{2}\alpha^2[m + k + b + \tfrac{1}{2}]^{-2} \qquad e_{m',k'} = 2m' + k' + b' + 1 \tag{24}$$

where $2b = (N-1)(1+\lambda N)-1$, $2b' = (N'-1)(1+\lambda' N')-1$, $g = \lambda(\lambda-1)$ and $g' = \lambda'(\lambda'-1)$. Now observe that equations (19) and (24) are consistent with each other provided the first, the third and the fourth relations of equation (23) are valid and, furthermore, the following relation between λ and λ' holds true:

$$\lambda' = \frac{N}{2N - 3}\lambda. \tag{25}$$

(c) Finally, consider the rational CSM of B_n type and the corresponding coulomb-like problem (see appendix A). The energy eigenvalues $E_{m,k}$ and $e_{m',k'}$ corresponding to these two cases are

$$E_{m,k} = -\tfrac{1}{2}\alpha^2[m + 2k + b + \tfrac{1}{2}]^{-2} \qquad e_{m',k'} = 2(m' + k') + b' + 1 \tag{26}$$

where $2b = (N-1)(1+2\lambda N)+2\lambda_1 N - 1$ and $2b' = (N'-1)(1+2\lambda' N')+2\lambda_1' N' - 1$. Again, it follows that equations (19) and (26) are consistent with each other provided the first, third and the fourth relations of equations (23) are valid and, furthermore, the following relation between the λ holds true:

$$\lambda_1' + (2N-3)\lambda' - \lambda_1 \frac{N}{N-1} - \lambda N = 0. \tag{27}$$

Note that equation (27) is satisfied provided $\lambda_1' = (N/(N-1))\lambda_1$ and λ and λ' are related as in the previous case, i.e. by equation (25). It may be noted here that for the D_N case $\lambda_1 = \lambda_1' = 0$, and hence in this case the relation (27) reduces to (25).

Summarizing, we find that for all types of CSM models in one dimension the mapping between the oscillator and the coulomb-like N-body problems holds good provided the first, third and the last relations of equation (23) are valid. It is worth pointing out that the first relation of equation (23) is also dictated by the coordinate transformation and is independent of the particular from of $V(x)$, as will be seen in the next section. It is amusing to note that the third and the fourth relations of equation (23) are also true for the usual SHO and the coulomb problems [6]. Thus, these must be universal relations valid for any $V(x)$ since these relations are also valid in the limit of vanishing $V(x)$. Note, however, that the relation between λ and λ' is dependent on the particular form of $V(x)$. Finally, it seems that relation (25) is universal in some sense for the mapping between the rational CSM of all types and the corresponding coulomb-like problems.

3. Coordinate transformation

In this section, we will be discussing the explicit coordinate transformation relating the CSM with the oscillator and the coulomb-like potentials. On comparing equations (6) and (15), we have the following operator relations

$$x = \frac{1}{2}y^2 \tag{28}$$

$$\frac{N-1}{2} + \sum_i x_i \frac{\partial}{\partial x_i} = \frac{1}{4}\left[N' + 2\sum_\mu y_\mu \frac{\partial}{\partial y_\mu}\right] \tag{29}$$

$$x\triangle_x - 2xV(\{x_i\}) = \frac{1}{2}[\triangle_y - 2V(\{y_\mu\})]. \tag{30}$$

Let us now assume a coordinate transformation of the form

$$x_i = f_i(\{y_\mu\}) \tag{31}$$

where the f_is are N arbitrary functions of the coordinates y_μ with the constraint $\sqrt{\sum_i f_i^2} = \frac{1}{2}y^2$. The particular form as well as the properties of all the f_j will be determined from equations (28)–(30). On multiplying both sides of equation (29) by x_j from the right and using relation (31), we encounter two different cases.

(a) $$N' = 2(N+1) \qquad \sum_\mu y_\mu \frac{\partial f_i}{\partial y_\mu} = 0. \tag{32}$$

However, the second relation of equation (32) implies that all the f_i are homogeneous functions of degree zero which is in direct contradiction with equation (28). Thus, this possibility is ruled out.

(b)
$$N' = 2(N + 1 - d) \qquad \sum_{\mu} y_\mu \frac{\partial f_i}{\partial y_\mu} = d f_i. \tag{33}$$

The second relation of equation (33) implies that all the f_i are homogeneous functions of degree d. However, it follows from equation (28) that d must be 2 and hence the first relation of (33) now reads

$$N' = 2(N - 1). \tag{34}$$

Equation (34) establishes a relationship between the total number of particles in the two cases. Notice that (34) also follows from a comparison of the eigenvalues in the two cases (see (23). It is amusing to note that exactly the same relation is also obtained in the case when one considers the mapping between the usual N-dimensional coulomb and N'-dimensional harmonic oscillator problems. In other words, (34) is independent of the particular form of the many-particle potential.

On multiplying both sides of equation (30) by x_j from the right and using relation (31) we obtain

$$[\triangle_y - 2V(\{y_\mu\})] f_i(\{y_\mu\}) + 2y^2 V(\{f_j\}) f_i(\{y_\mu\}) = 0. \tag{35}$$

This is a set of highly nonlinear second-order differential equations. Moreover, only those solutions for which all the f_i are homogeneous functions of degree 2 and the norm of the f_i is $\frac{1}{\sqrt{2}} y$ are acceptable solutions for our purpose.

One would now like to ask if such a solution (to (35)) exists or not. Note at this point that for acceptable solutions the first term of (35) (i.e. $L_i = \triangle_y f_i$) should either be a constant or be a homogeneous function of degree zero. Let us first consider the case $L_i = 0$, i.e. those solutions which are also solutions of the N'-dimensional Laplace equation. With the use of (35), this implies the following relation between $V(x)$ and $V(y)$,

$$V(x) = y^{-2} V(y). \tag{36}$$

In this case the operator relations (28)–(30) are identical to those in the case of the usual H_{sho} and H_{c} problems. Now exactly following the procedure as given in Zeng *et al* [6], we find one valid coordinate transformation between the two problems as given by

$$x_i = f_i = \frac{1}{4} \sum_{\alpha,\beta} \Gamma^i_{\alpha\beta} y_\alpha y_\beta \tag{37}$$

where the matrices Γ constitute the Clifford algebra,

$$\Gamma^i \Gamma^j + \Gamma^j \Gamma^i = 2\delta^{ij}. \tag{38}$$

We might add here that the coordinate transformation (37) can be written down explicitly with the use of the real representation of the Clifford algebra [8]. However, we have to determine the form of $V(x)$ such that equation (36) is consistent with the coordinate transformations as given by (37) and (38). One such choice is $V(x) = 4gx^{-2}$ and $V(y) = gy^{-2}$. We may add here that unfortunately none of the inverse square interactions of the CSM satisfy (36).

Let us now consider the second possibility, i.e. all the L_i are non-zero arbitrary constants. In this case the x_i are not independent of each other and no valid solution can be found. Thus, it seems that the L_i as homogeneous functions of degree zero is probably the only alternative for finding an explicit coordinate transformation in the interesting case of the CSM. However, finding such solutions explicitly or even proving the existence of such solutions is a highly non-trivial problem and at present we do not have any answer to this question.

Finally, as an aside, let us note that the mapping between H_c and H_{sho} as described by equations (18) and (19) is valid even when the many-body interaction of the two problems is not the same (i.e. they have a completely different functional dependence). However, both should satisfy the homogeneity condition (3). In such cases let us denote $V(y)$ by $\tilde{V}(y)$. Now note that the coordinate transformation (37) can be identified as the required coordinate transformation provided it relates $V(x)$ and $\tilde{V}(y)$ as follows:

$$V(x) = y^{-2}\tilde{V}(y) \qquad \tilde{V}(y) = 2xV(x). \tag{39}$$

Thus, with each type of rational CSM one can associate a new many-body problem with coulomb-like interaction which is related by the coordinate transformation (37). Similarly, one can find new many-body Hamiltonians with oscillator confinement associated with H_c. In particular, H_c with $V(x)$ given by (4) is related to H_{sho} with $\tilde{V}(y)$ given by

$$\tilde{V}(y) = 8gy^2 \sum_{i<j} \left[\sum_{\alpha\beta} (\Gamma^i - \Gamma^j)_{\alpha\beta} y_\alpha y_\beta \right]^{-2} \tag{40}$$

where $(\Gamma_i - \Gamma_j)_{\alpha\beta}$ implies the $\alpha\beta$ element of the matrix $\Gamma_i - \Gamma_j$. Note that, for the real representation of the Clifford algebra [8], some of the Γ's are diagonal and, consequently, the many-body interaction (40) is an $N'(= 2(N-1))$-body interaction unlike in the case of the usual CSM. This type of new many-body Hamiltonian may or may not be interesting from the physical point of view. However, they have the remarkable property of being exactly solvable.

4. The mapping: higher-dimensional generalization

In the last two sections we have established the mapping between the oscillator and the coulomb-like problem in one-dimensional many-body systems. We now generalize these results to higher-dimensional many-body systems with the many-body interactions as homogeneous functions of degree -2. Recall at this point that the many-body interaction of all the known higher-dimensional CSM type models is homogeneous with degree -2. For example, the Calogero–Marchioro model [9], models with novel correlations [10], models with two-body interactions [11] and models considered in [12, 13] have this property.

Let us consider the operators K_1, K_2 and K_3 for the Coulomb-like problem as follows

$$\begin{aligned} K_1 &= \frac{1}{2}(X\triangle_X - 2XV(X) + X) \\ K_2 &= \mathrm{i}\left(\frac{ND-1}{2} + \sum_i \vec{r}_i \cdot \vec{\nabla}_i\right) \\ K_3 &= -\frac{1}{2}(X\triangle_X - 2XV(X) - X) \end{aligned} \tag{41}$$

where

$$X = \sqrt{\sum r_i^2} \qquad \triangle_X = \sum_i \nabla_i^2 \tag{42}$$

and $\vec{\nabla}_i$ is the D-dimensional gradient of the ith particle. The potential $V(X)$ is homogeneous with degree -2 and satisfies the homogeneity condition analogous to equation (3). One can check that these three operators constitute the $SU(1,1)$ algebra (7). The eigenequation of the Hamiltonian

$$H_c^D = -\frac{1}{2}\triangle_X + V(X) - \frac{\alpha}{X} \tag{43}$$

can be shown to be given by equation (10) with k_2 and k_3 replaced by K_2 and K_3, respectively.

Similar to the one-dimensional oscillator problem, we define the three operators for the D'-dimensional many-body problem with an oscillator potential as [12]

$$K_1 = \frac{1}{4}(\triangle_Y + Y^2 - 2V(Y))$$
$$K_2 = \frac{\mathrm{i}}{4}\left(N'D' + 2\sum_\mu \vec{r}'_\mu \cdot \vec{\nabla}'_\mu\right)$$
$$K_3 = \frac{1}{2}H^{D'}_{\mathrm{sho}} = \frac{1}{4}(-\triangle_Y + Y^2 + 2V(Y)) \tag{44}$$

where $\triangle_Y$ and Y^2 are given as

$$\triangle_Y = \sum_\mu \nabla'^2_\mu \qquad Y = \sqrt{\sum_\mu r'^2_i}. \tag{45}$$

We denote ∇'_μ as the D'-dimensional gradient operator for the μth particle. These three operators satisfy the $SU(1,1)$ algebra (7) and the Hamiltonian is proportional to K_3.

Following the discussions of section 2.3, one can establish the mapping between the eigenvalues as well as the eigenvector of H^D_{c} and the same quantities of $H^{D'}_{\mathrm{sho}}$. Equations (18) and (19) continue to be valid in the higher-dimensional case also but with k_2 replaced by K_2. In particular,

$$|N, D, M\rangle = \mathrm{e}^{-\mathrm{i}K_2\theta_M}|N', D', M'\rangle \qquad E_M = -\frac{2\alpha^2}{(e_{M'})^2}. \tag{46}$$

An analysis of equations (41), (42), (44) and (45), on the lines of what has been done in the previous section, shows that the relation

$$N'D' = 2(ND - 1) \tag{47}$$

holds true for any $V(X)$. Note that this equation reduces to (34) for $D = D' = 1$.

We would like to emphasize here that unlike the one-dimensional case, the higher-dimensional many-body systems, like the Calogero–Marchioro model [9] or the models for novel correlations [10], have part of the energy spectrum with a linear dependence and the remaining part with a nonlinear dependence on the coupling constant of the relevant problem. Unfortunately, so far, only the linear part of the spectrum has been obtained analytically for all the known higher-dimensional many-body problem. In fact, not even one energy level with nonlinear dependence on the coupling constant has been obtained as yet. Not surprisingly, even using the underlying $SU(1,1)$ symmetry of the Calogero–Marchioro problem, one cannot find the missing nonlinear part [12]. This is because the angular part of the Hamiltonian or equivalently the eigenvalue problem of the Casimir operator cannot be solved exactly in higher dimensions. Thus, we are unable to compare the energy spectra of H^D_{c} and $H^{D'}_{\mathrm{sho}}$ in higher dimensions as has been done for the one-dimensional systems.

5. Summary

In this paper we have shown that the energy spectrum as well as the eigenfunctions of the rational CSM with a Coulomb-like interaction associated with the root structure of A_N, B_N, C_N, D_N and BC_N can be obtained from the corresponding CSM with the harmonic oscillator potential. Consequently, all types of CSM with a coulomb-like interaction are also exactly solvable models. Thus, one has added a new class of members to the family of exactly solvable many-body systems in one dimension. Furthermore, we have shown that all these results can be generalized to other many-body systems in one dimension provided that the many-particle interaction of these systems, much akin to the CSM, is a homogeneous function of degree -2. We have explicitly found the coordinate transformation for some specific cases which maps the coulomb-like problem to a harmonic one. Although we are not able to find the coordinate transformation responsible for such a mapping for each and every case, we have found a set of second-order coupled nonlinear differential equations and shown that a particular class of solutions of this set of equations are going to determine the coordinate transformation. However, the proof of the existence of such a class of solutions and, if possible, how to find them explicitly is a highly non-trivial problem.

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Appendix A. B_N CSM with coulomb-like potential

In this appendix we obtain the spectrum as well as the eigenfunctions of the B_n type CSM with coulomb-like potential. In particular, we consider the Hamiltonian (1) with $V(x)$ given by (5) and $g_1 = \lambda(\lambda - 1)$, $g_2 = \lambda_1(\lambda_1 - 1)$ and $g_3 = 0$. Note that the energy eigenstates of the BC_N as well as the C_N CSM could be obtained easily from the known results of the B_N CSM. Let

$$\Phi = \prod_l x_l^{\lambda_1} \prod_{i<j} (x_i^2 - x_j^2)^\lambda P_{2k}(x)\phi(x) \tag{A1}$$

be a solution of the Schrödinger equation $H_c\Phi = E\Phi$. In equation (A1), $P_{2k}(x)$ is a symmetric homogeneous polynomial of the coordinates with degree $2k$ and satisfies the generalized Laplace equation,

$$\triangle_x P_{2k}(x) + 2\lambda_1 \sum_i x_i^{-1} \frac{\partial P_{2k}}{\partial x_i} + 4\lambda \sum_{i\neq j} \frac{x_i}{x_i^2 - x_j^2} \frac{\partial P_{2k}}{\partial x_i} = 0. \tag{A2}$$

Plugging the expression (A1) into the Schrödinger equation, we have

$$\phi'' + [2b + 4k + 1]\frac{\phi'}{x} + 2\left(E + \frac{\alpha}{x}\right)\phi = 0 \tag{A3}$$

where the parameter b is given by

$$b = \tfrac{1}{2}(N-1)(1 + 2\lambda N) - \tfrac{1}{2} + \lambda_1 N. \tag{A4}$$

Defining a new variable $t = \sqrt{2E}x$, equation (A3) can be solved as

$$\phi_{n,k} = \exp(-t) L_n^{2b+4k}(2t) \tag{A5}$$

where $L_n^{2b+4k}(2t)$ is the Laguerre polynomial with argument $2t$. The energy eigenvalues corresponding to the wavefunctions (A1) are

$$E_{n,k} = -\tfrac{1}{2}\alpha^2[n + 2k + \tfrac{1}{2} + b]^{-2}. \tag{A6}$$

It may be noted here that the results for the D_N case can be obtained from here simply by putting $\lambda_1 = 0$.

The wavefunction given by (A1) contains a homogeneous function P_{2k} of degree $2k$ which is determined by equation (A2). In general, we do not know the exact solutions of equation (A2) except for some small values of N and k. However, it can be shown that equation (A2) is exactly solvable by following the methods described in Brink *et al* [14]. In fact, apart from some constant, the corresponding equation in [14] contains one more extra term $\sum_i x_i \partial/\partial x_i$ than (A2) and the treatment as well as conclusions obtained there are also valid in the case of equation (A2).

Appendix B. Casimir operator and separation of variables

In this appendix we study the role of the Casimir operator of the $SU(1,1)$ group regarding the separation of variables in the case of the Schrödinger equation for $H_{\rm c}$ and $H_{\rm sho}$. The Casimir operator of $SU(1,1)$ for the class of unitary irreducible representations, called the positive discrete series, is defined by [7, 12]

$$C = k_3^2 - k_1^2 - k_2^2 \tag{B1}$$

and it commutes with all the generators k_1, k_2 and k_3. The Casimir operator and k_3 are diagonal in this representation and the eigenvalue of k_3 is given by

$$\epsilon_\pm = n + \tfrac{1}{2} \pm (q + \tfrac{1}{4})^{1/2} \tag{B2}$$

where n is a non-negative integer and q is the eigenvalue of the Casimir operator. ϵ_- has the restriction $(q + \frac{1}{4})^{1/2} < \frac{1}{2}$ and it leads to physically unacceptable solutions [12]. Thus, we will be concerned with ϵ_+ only in this paper.

We use the notation C_N^x and $C_{N'}^y$ for the Casimir operators associated with the generators of $SU(1,1)$ given by two different representations (6) and (15), respectively. Plugging (6) and (15) into (B1) and after some manipulation [12], we find

$$\begin{aligned} C_N^x &= \frac{1}{4}(N-1)(N-3) + 2x^2 V(x) - \sum_{i<k}\left(x_i \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_i}\right)^2 \\ C_{N'}^y &= \frac{1}{16}N'(N'-4) + \frac{1}{2}y^2 V(y) - \frac{1}{4}\sum_{\mu<\nu}\left(y_\nu \frac{\partial}{\partial y_\mu} - y_\mu \frac{\partial}{\partial y_\nu}\right)^2. \end{aligned} \tag{B3}$$

Now since $V(x)$ is homogeneous with degree -2, hence $x^2 V(x)$ can be expressed purely in terms of the $N-1$ angular variables in N-dimensional spherical coordinates. Similarly, $y^2 V(y)$ is determined solely in terms of the $N'-1$ angular variables of N'-dimensional spherical coordinates. Thus, apart from a constant factor, both C_N^x and $C_{N'}^y$ are exactly equivalent to the angular part of the respective Hamiltonians $H_{\rm c}$ and $H_{\rm sho}$. In particular, the angular part of the Hamiltonians $H_{\rm c}$ and $H_{\rm sho}$ is given by

$$H_{\rm c}^a = C_N^x - \tfrac{1}{4}(N-1)(N-3) \qquad H_{\rm sho}^a = C_{N'}^y - \tfrac{1}{16}N'(N'-4). \tag{B4}$$

Furthermore, the constant factor of C_N^x is related to the constant factor of $C_{N'}^y$ by (34). It may be noted here that the total angular momenta L^2 and L'^2,

$$L^2 = -\sum_{i<k}\left(x_i \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_i}\right)^2 \qquad L'^2 = -\sum_{\mu<\nu}\left(y_\nu \frac{\partial}{\partial y_\mu} - y_\mu \frac{\partial}{\partial y_\nu}\right)^2 \tag{B5}$$

of $H_{\rm c}$ and $H_{\rm sho}$, respectively, are also related to each other as

$$L'^2 = 4L^2 \qquad l' = 2l \tag{B6}$$

in the case when relation (36) is satisfied. In equation (B6), l and l' denote the eigenvalues of L and L', respectively. This result is also valid in the case when one starts with $\tilde{V}(y)$ instead of $V(y)$ in $H_{\rm sho}$ and relation (39) holds true.

Following Gambardella [12], it is easily seen that the relation $[C, k_3] = 0$ implies

$$[H_{\rm c}^r, H_{\rm c}^a] = 0 \qquad [H_{\rm sho}^r, H_{\rm sho}^a] = 0 \tag{B7}$$

where H_c^r and H_{sho}^r are the radial part of the N-dimensional conventional coulomb problem and the N'-dimensional conventional oscillator problems, respectively. We have used the relation

$$k_3 = \alpha + \tfrac{1}{2}x + xH_c \qquad \text{(B8)}$$

in order to derive the first equation of (B7). The relations (B7) imply that the method of separation of variables is applicable to both H_c and H_{sho}. This relation for H_{sho} in arbitrary dimensions was known earlier [12], while we have generalized this result to the case of H_c.

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